বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2021

(Under CBCS Pattern)

Semester - III

Subject : MATHEMATICS

Paper : GE 3 - T

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

[DIFFERENTIAL EQUATION AND VECTOR CALCULUS]

(Theory)

Group - A

1. Answer any ***four*** of the following questions : 12×4=48

(i) (a) Solve by the method of undetermined coefficients

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = (x-2)e^x$$

(b) Evaluate : $\frac{1}{D^4 + 2D^3 - 3D^2}(4\sin x)$ 8+4

(ii) (a) Define first order linear ordinary differential equation. Find the integrating factor of the differential equation $\cos x \frac{dy}{dx} + y \sin x = 1$. Solve the differential equation

$$\frac{dy}{dx} + \frac{1}{1+x^2} y = \frac{e^{tan^{-1} x}}{1+x^2}.$$

(b) Let us consider the equation $\frac{d^2y}{dx^2} + y = 0$ and find a series solution of this equation near the ordinary point. (1+2+4)+5

(iii) (a) Solve :

$$\frac{dx}{dt} - 7x + y = 0$$

$$\frac{dy}{dt} - 2x - 5y = 0$$

(b) Solve the equation $\frac{dx}{dt} = -wy$ and $\frac{dy}{dt} = wx$ and show that the point (x, y) lies on a circle. 8+4

(iv) (a) Solve : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^{3x}$.

(b) Solve the simultaneous equation :

$$\frac{adx}{yz(b-c)} = \frac{bdy}{zx(c-a)} = \frac{cdz}{xy(a-b)}$$

6+6

(v) (a) Use Picard's Method to compute approximately the value of y when x = 0.1 from the initial value problem $\frac{dy}{dx} = x + y$, when y(0)=1. Check the result with the exact value.

(b) Determine the steady state and their stability of the differential equation $\dot{y} = f(y) = y^2 - 5y + 6$ 8+4

(vi) (a) Solve $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 3y = 2\sec x$, when *sec x* being a solution.

(b) Solve by the method of variation of parameter, the equation $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$.

6+6

(vii) (a) Show that the volume of the parallelopiped, whose edges are represented by $(3i+2j-4k)$, $(3i+j+3k)$ and $(i-2j+k)$ is 49 cubic units.

(b) Let α, β, γ be three vectors. Reduce the expression

$(\beta+\gamma).\{(\gamma+\alpha)\times(\alpha+\beta)\}$ to its simplest form and prove that it vanished when α, β, γ are coplanar.

(c) Let *a, b, c* be three vectors. Prove the identity $[a\times b, b\times c, c\times a] = [a\,b\,c]^2$.

4+4+4

(viii) (a) State the Green's theorem. Using this theorem evaluate the integral $\oint x^2dx + (x+y^2)dy$ along the curve $C: y=0, y=x$ and $y^2 = 2-x$ in the first quadrant.

(b) Show that $\vec{F} = (2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential. (1+5)+(2+4)

Group - B

2. Answer any ***six*** of the following questions : 2×6=12

(i) Find the steady state point (or Equilibrium point) for the system of equations $\dot{x} = x + 2y$ and $\dot{y} = x^2 + y$.

(ii) Show that the equation $\frac{dy}{dx} = 3xy^{\frac{1}{3}}, y(0) = 0$ has no unique solution.

(iii) Using Wronskian, show that $e^x, \cos x, \sin x$ are linearly independent.

(iv) Show that $x\left(y^2-a^2\right)dx+y\left(x^2-z^2\right)dy-z\left(y^2-a^2\right)dz$ is integrable.

(v) Show that $x=0$ is a regular singular point of $\left(2x+x^3\right)y''-y'-6xy=0$.

(vi) Solve $\frac{d^2x}{dt^2}+n^2x=0$, when $t=0$, $\frac{dx}{dt}=0$ and $x=0$.

(vii) Define regular singular point and irregular singular point of an ordinary differential equation $P_0(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0$.

(viii) Prove that if $\vec{a},\vec{b},\vec{c}$ are coplanar then $\left[\vec{a}\vec{b}\vec{c}\right]=0$.

(ix) Evaluate $\int_C F.dr$ where $F=x^2y^2i+yj$ and the curve C is $y^2=4x$ in the xy – plane from (0, 0) to (4, 4).

(x) Show that the vector $a=\left(x^2-yz\right)i+\left(y^2-zx\right)j+\left(z^2-xy\right)k$ is an irrotational vector.

OR

[GROUP THEORY-I]

(Theory)

Group - A

1. Answer any ***four*** of the following questions : 12×4=48

(a) (i) Define abelian group. Show that the eight symmetries of the square form a non abelian group. 2+4

(ii) In a group $(G, \circ)$, prove that

(i) $(a \circ b)^{-1} = b^{-1} \circ a^{-1}, \forall\ a, b \in G$

(ii) $a \circ b = a \circ c \Rightarrow b = c$ 4+2

(b) (i) Prove that the set $H = \left\{ \begin{vmatrix} a & b \\ 2a & b \end{vmatrix} : a, b \in R \text{ and } a^2 + b^2 = 1 \right\}$ forms a commutative group with respect to matrix multiplication. 6

(ii) Show that the set of all permutations on a set of three elements is a non-abelian group. 6

(c) (i) Let $(G, \circ)$ be a group and $a \in G$. Prove that $Z(G)$, the centre of the group, is a subgroup of $C(a)$, the centralizer of a. 6

(ii) Let G be a group in which $(ab)^3 = a^3 b^3, \forall a, b \in G$. Show that $H = \{x^2 : x \in G\}$ is a subgroup of G. 6

(d) (i) Let $G = S_3, G' = (\{1, -1\}, \bullet)$ and $\phi : G \to G'$ be defined by

$$\phi(\alpha) = \begin{cases} 1, \text{if } \alpha \text{ be an even permutation in } S_3 \\ -1, \text{if } \alpha \text{ be an odd permutation in } S_3 \end{cases}$$

Prove that ϕ is a homomorphism. 6

(ii) State and prove first isomorphism theorem. 6

(e) (i) Define a cyclic group. Find the generators of U(10). 3

(ii) Show that in a group G the right and left cancellation law hold. Is the converse true? Justify your answer. 3

(iii) Prove that if the identity permutation $\alpha = \beta_1 \beta_2 \ldots\ldots \beta_r$ where the β's are 2-cycles, then r is even. 6

(f) (i) If G is a group with more than one element and G has no proper, nontrivial subgroup. Then prove that G is a finite group of prime order. 6

(ii) Let H be a subgroup of G and K be a nontrivial subgroup of G. Prove that $HK / K \approx H / H \cap K$. 6

(g) (i) Let G be an Abelian group with identity e. Prove that $H = \{x \in G \mid x^2 = e\}$ is a subgroup of G. 3

(ii) Let a and b be elements of a group. If $|a| = 10$ and $|b| = 21$, show that $\langle a \rangle \cap \langle b \rangle = (e\}$. 3

(iii) Determine all homomorphisms from Z_{12} to Z_{30}. 6

(h) (i) Find the homomorphic images of S_3. 3

(ii) Define Centralizer $C(G)$ of a group G. Is $C(G)$ Abelian? 3

(iii) Let $G = \langle a \rangle$ be a cyclic group of order n. Show that $G = \langle a^k \rangle$ if and only if $gcd(k, n) = 1$. 6

Group - B

2. Answer any ***six*** of the following questions : 2×6=12

(a) In a group $(G, \circ)$, a is an element of order 30. Find the order of a^{18}.

(b) Find the order of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix}$.

(c) Find the number of generators of the cyclic group $(S, \bullet)$ where $S = \{1, -1, i, -i\}$

(d) If $G = S_3$ and $H = A_3$ then find $[G : H]$.

(e) If $G = (Z, +)$ and $H = (3Z, +)$, then find the distinct left cosets of H.

(f) Let $(G, \circ)$ and $(G', *)$ be two groups and $\phi : G \to G'$ be a homomorphism. Show that $\phi(a^{-1}) = \{\phi(a)\}^{-1}, \forall a \in G$.

(g) Every group of order 35 is cyclic or not? Justify your answer.

(h) Consider the group Z_{30}. Find the smallest positive integer n such that $n[5] = [0]$ in Z_{30}.

(i) Give an example of a group such that normality is not transitive.

(j) Let H is a subgroup of G and $g_1, g_2 \in G$. Then $Hg_1 = Hg_2$ if and only if $g_1^{-1}H = g_2^{-1}H$.

OR

[THEORY OF REAL FUNCTIONS AND INTRODUCTION TO METRIC SPACE]

(Theory)

Group - A

1. Answer any ***four*** of the following questions : 12×4=48

(a) (i) Show that the limit $\lim_{x\to 0}\sin\left(\frac{1}{x^2}\right)$ does not exist.

(ii) Let $D \subset R$ and $f : D \to R$ be a function. Let $c \in D \cap D'$. Show that f is continuous at c if and only for every sequence $\{x_n\}$ in D converging to c.

(iii) A function $f : R \to R$ is defined by $f(x)=\begin{cases} 2x, & x\in Q \\ 1-x, & x\in R-Q \end{cases}$. Prove that f is continuous at $\frac{1}{3}$ and discontinuous at every other point. 3+5+4

(b) (i) Prove $I = [a, b]$ be a closed and bounded interval and a function $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.

(ii) Verify Rolle's theorem of the function $f(x) = x^2 + \cos x$ on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

(iii) Prove that $\frac{2x}{\pi} < \sin x$ for $0 < x < \frac{\pi}{2}$. 5+4+3

(c) (i) State the prove that Cauchy's Mean Value Theorem.

(ii) A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x)=\begin{cases} 0, & \text{if } x = 0 \text{ or } x \text{ is irrational.} \\ \frac{1}{q^3} \text{ if } x=\frac{p}{q}, & \text{where } p \in \mathbb{Z},\ q \in \mathbb{N}, \text{ and } gcd\,(p, q) = 1. \end{cases}$$

Show that f is differentiable at $x = 0$ and $f'(0) = 0$.

(iii) If $k \in \mathbb{R}$ and $f(x) = k$, $x \in \mathbb{R}$. Find the derived function f' and its domain.

5+5+2

(d) (i) Let $f : R \to R$ be such that $f(x+y) = f(x) + f(y)$ for all x, y in R. If $\lim_{x \to 0} f = L$ exists, then show that $L = 0$.

(ii) Give examples of functions f and g such that f and g do not have limits at a point c, but such that both $f + g$ and fg have limits at c.

(iii) If f is uniformly continuous on $A \subset R$, and $|f(x)| > k > 0$ for all $x \in A$, show that $\frac{1}{f}$ is uniformly continuous on A. 5+3+4

(e) (i) If (X, d) is a metric space then prove that finite intersection of open sets in X is open.

(ii) Let (X, d) is a metric space and $A \subset X$. Then prove that A is closed iff A contains all its limit points.

(iii) Let (Y, d_Y) be a subspace of a metric space (X, d) and $A \subset Y$. Then prove that $x \in Y$ is a limit point of A in Y iff x is a limit point of A in X.

4+4+4

(f) (i) Show that (R^2, d) is a metric space, where the mapping $d : R^2 \times R^2 \to R$ is given by $d(x,y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}, (x_1, y_1), (x_2, y_2) \in R^2$.

(ii) Prove that a finite set has no limit points.

(iii) If (X, d) is a metric space then prove that a set $A \subset X$ is closed in (X, d) iff its complement $A' = X \setminus A$ is open in (X, d). 5+3+4

(g) (i) Use Taylor theorem to prove that $1 + \frac{x}{2} - \frac{x^3}{8} < \sqrt{1+x} < 1 + \frac{x}{2}$ if $x > 0$.

(ii) Let $f : R \to R$ be a differentiable function on R and $f'(x) > f(x)$ for all $x \in R$. If $f(0) = 0$, prove that $f(x) > 0$ for all $x > 0$. 6+6

(h) (i) In the set R^2 of all order pairs of real numbers let us define a function $d : R^2 \times R^2 \to R$ by $d(x,y) = |x_1 - y_1|$ where $x = (x_1, x_2), y = (y_1, y_2) \in R^2$. Show that d is a pseudo-metric on R^2.

(ii) Let (X, d) be any metric space. Is $(X, \bar{d})$ a Metric space where $\bar{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}$. 6+6

Group - B

2. Answer any ***six*** of the following questions : 2×6=12

(a) From definition of limit show that $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$. Find δ if ε = 0.005.

(b) Give an example of a function which is differentiable at a point but the derivative is not continuous at the same point.

(c) Give an example of a uniformly continuous function on [0, 1] that is differentiable on (0, 1) but whose derivative is not bounded on (0, 1).

(d) Show that the function $f(x) = x + [x]$ is piecewise continuous in [0,2].

(e) If $p(x)$ be a polynomial of degree >1 and $k \in R$, then prove that between any two real roots of $p(x) = 0$ there is a real root of $p'(x) + kp(x) = 0$.

(f) The set of all limit points of a bounded sequence is bounded.

(g) Show that union of arbitrary number of closed sets is a closed set.

(h) Give an example of an incomplete metric space.

(i) Show that every open sphere is an open set but not conversely.

(j) Show that $x < \tan x$ in $0 < x < \frac{\pi}{2}$.